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ISOTHERMIC SHOCKS IN ELECTROHYDRODYNAMICS

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One-dimensional stationary electrohydrodynamic flows of a medium are investigated in connection with the problem of shock wave structure with allowance for thermal conductivity effects when those of viscosity can be neglected. The behavior of integral curves of equations which describe such flow is investigated in the velocity-electric (uE) and in the velocity-temperature (uT) fields. Conditions for which a continuous shock wave structure, produced by the thermal conductivity mechanism, exists, are determined. It is shown that it is possible for the shock wave to have a structure with an isothermic shock, while at the same time it is impossible to obtain a continuous flow inside that structure. Conditions for which such cases are possible are established. Depending on conditions ahead of the wave front, the electric field may be either constant or alternating. Relationships which make it possible to determine the field downstream of the wave front are indicated for the latter case.

1. Statement of the problem. Let us consider the flow in an electric field of an inviscid heat-conducting gas with a positive volume charge $q > 0$. We select the system of coordinates with the x-axis directed downstream. We assume that all quantities depend only on x, and that the gas velocity, the electric field and the density of the electric current have components only along the x-axis. We denote these quantities, respectively, by u, E and j. In an electrohydrodynamic approximation the considered flow is defined by the following system of equations [1]:

$$\rho^* u^* = m, \quad p^* = \rho^* R T^*, \quad \rho^* u^{*2} + p^* - \frac{E^{*2}}{8\pi} = \Pi^* \tag{1.1}$$

$$\lambda \frac{dT^*}{dx} = \rho^* u^* \left(c_p T^* + \frac{1}{2} u^{*2}\right) + j(\varphi^* - \varphi_1^*) - \Sigma^*$$

$$\frac{d\varphi^*}{dx} = -E^*, \quad \frac{dE^*}{dx} = 4\pi q^*, \quad q^*(u^* + bE^*) = j$$

where m, Σ^*, Π^* and j are constants of integration determined by flow parameters at some point $x = x_1$, where they are denoted by subscript 1.

Investigation of the electrohydrodynamic shock waves with allowance for thermal conductivity will be carried out in the formulation adopted in [2, 3]. For $q^* \neq 0$ the considered electrohydrodynamic flow is everywhere nonuniform. It is assumed that a zone $\Gamma^* = \{x_2 \leqslant x \leqslant x_3\}$ of considerable gradients of some of the flow parameters, as compared to gradients outside it, exists in the flow region.

We introduce quantities $l = \lambda / (c_p \rho_1^* u_1^*)$ and $L = u_1^{*2} / (4\pi b \mid j \mid)$ whose dimension is that of length, and pass to the dimensionless variables

$$\rho = \frac{\rho^*}{\rho_1^*}, \quad u = \frac{u^*}{u_1^*}, \quad T = \frac{T^*}{T_1^*}, \quad E = \frac{bE^*}{u_1^*}, \quad q = \frac{q^* u_1^*}{|j|}, \quad \varphi = \frac{b\varphi^*}{L u_1^*}$$

$$\gamma = \frac{c_p}{c_v}, \quad M_1 = \frac{u_1^*}{(\gamma R T_1^*)^{1/2}}, \quad S = \frac{1}{8\pi\rho_1^* b^2}$$

$$J = \frac{j}{|j|}, \quad \varepsilon = \frac{l}{L}, \quad \zeta = \frac{x}{L}$$

If the coefficients of thermal conductivity λ and mobility b are constant, the system of Eqs. (1.1) in these variables assumes the form

$$\rho u = 1, \quad u + \frac{1}{\gamma M_1^2}\frac{T}{u} - SE^2 = \Pi \tag{1.2}$$

$$\frac{\varepsilon}{(\gamma-1)M_1^2}\frac{dT}{d\zeta} = \frac{1}{(\gamma-1)M_1^2}T + \frac{1}{2}u^2 + 2SJ(\varphi - \varphi_1) - \Sigma \tag{1.3}$$

$$\frac{d\varphi}{d\zeta} = -E, \quad \frac{dE}{d\zeta} = q, \quad q = \frac{J}{u+E} \tag{1.4}$$

$$\Pi = 1 + \frac{1}{\gamma M_1^2} - SE_1^2, \quad \Sigma = \frac{1}{(\gamma-1)M_1^2} + \frac{1}{2} - \alpha,$$

$$\alpha = \frac{\varepsilon}{(\gamma-1)M_1^2}\frac{dT}{d\zeta}\Big|_{\zeta=\zeta_1}$$

Let us establish the physical meaning of quantities l and L of dimension of length. The thermal conductivity coefficient $\lambda \sim R\rho_1^*\tau$ [1], where τ is the time of gas particle free run. Let l_* be the length of the free path and $u_t \sim (RT^*)^{1/2}$ the mean thermal velocity of particles. Then

$$l \sim \tau R T_1^* / u_1^* \sim l_* u_t / u_1^* \lesssim l_*$$

The length l is, thus, of the order of the mean free path of gas particles. The expression for length L can be rewritten as $L = E_0 / (4\pi q_0)$, where $E_0 = u_1^*/b$ and $q_0 = |j|/u_1^*$. The quantities E_0 and q_0 were used as the characteristic intensities of the electric field and charge density for reducing equations to dimensionless form. The sixth of Eqs. (1.1) shows that the length L is of the order of the length required for the charge q_0 to generate field E_0. In other words, the characteristic dimension of nonuniformity in the region outside the zone of considerable gradients is taken as the characteristic length L for reducing the variable x to dimensionless form.

We assume that the ratio of lengths $l / L = \varepsilon \ll 1$. Let furthermore the interaction parameter S be arbitrary, the dimensionless field $E_1 \ll \varepsilon^{-1}$, and the Mach number $M_1 > 1$.

It will become evident from the qualitative analysis of integral curves of system (1.2)–(1.4) given below that the length of the zone of considerable gradients $\Gamma\ \{\zeta_2 \leqslant \zeta \leqslant \zeta_3\}$ (zone of the shock wave structure) is of the order of ε. We choose points ζ_1 and ζ_4 outside the zone Γ close to points ζ_2 and ζ_3, respectively. From the first of Eqs. (1.4) we have $\varphi(\zeta_4) - \varphi(\zeta_1) \approx \Delta\zeta E_1 \sim \varepsilon E_1 \ll 1$, i.e. the potential variation in the region of the shock wave structure is small and it is, consequently, possible to neglect the penultimate term in the right-hand side of Eq. (1.3). We introduce the notation

$$L_1(u, E) = u + E, \quad L_2(u, E, \alpha) = \frac{\gamma+1}{2}u + \frac{(\gamma-1)\Sigma}{u} - \gamma(\Pi + SE^2)$$

$$L_3(u, E, \alpha, \varepsilon) = \left(\frac{1}{\gamma}L_1L_2 + 2\varepsilon JSE\right)u, \quad L_4(u, E) = \Pi + SE^2 - 2u$$

Taking into account the second of Eqs. (1.2) we transform Eq. (1.3) to

$$\frac{\varepsilon}{M_1^2}\frac{dT}{d\zeta} = -uL_2$$

2. Shock wave structure produced by the mechanism of heat conduction. Let us investigate the behavior of integral curves of Eqs.(1.2) — (1.4) in region $u > 0$ in the uE-plane. We shall show that under specific conditions an isothermic shock must occur in the model of medium defined by system (1.1).

From the second of Eqs.(1.2) we have

$$\frac{dT}{d\zeta} = \gamma M_1^2 \left(L_4 \frac{du}{d\zeta} + 2SuE\frac{dE}{d\zeta}\right) \qquad (2.1)$$

It follows from formulas (1.5) and (2.1) that at points of the uE-plane where $L_4 \neq 0$ the relationship

$$\varepsilon \frac{du}{d\zeta} = -\frac{L_3}{L_1 L_4} \qquad (2.2)$$

is satisfied. Using (2.2) and the last two of Eqs.(1.4) we obtain

$$\frac{dE}{du} = -\varepsilon J\frac{L_4}{L_3} \qquad (2.3)$$

Let us consider the case when $\Pi > 0$ and $\Sigma > 0$. We construct in the half-plane $u > 0$ the lines $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ and $L_4 = 0$ (Figs. 1, 4 — 7), which subsequently are denoted by L_1°, L_2°, L_3° and L_4°. We also construct curve $L_5^\circ(u, E)$ at which the current Mach number is unity. The condition $M_1^2u^2 = T$ must be satisfied at points lying on L_5°.

Taking into consideration the second of Eqs.(1.2) we obtain that L_5° is a parabola with its vertex at point A $(\gamma(\gamma+1)^{-1}\Pi, 0)$

$$L_5 \equiv u - \gamma(\gamma+1)^{-1}(\Pi + SE^2) = 0$$

It will be seen that for $u \to \infty$ parabola L_5° lies in the region comprised between parabola L_4° and curve L_2°. Note that the vertex of parabola L_4° lies for $\gamma > 1$ to the left of the vertex A of parabola L_5°.

For $\varepsilon \to 0$, i.e. when the pass to an ideal (non-heat-conducting) flow, it follows from (1.5) that in the region $u > 0$ the curve $L_2^\circ(u, E, 0)$ defines the relation between the electric field and the velocity in an ideal flow near the shock wave front, where variations of the potential can be neglected. Curve L_2° has a vertical asymptote $u = 0$ and two branches $L_{2\pm}^\circ$ which are symmetric about the u-axis and pass through points $(1, \pm(E_1 - \alpha(\gamma-1)/\gamma S)^{1/2})$. For fixed α, γ, M_1 and S the disposition of branches is determined by the intensity of the electric field at point ζ_1.

If

$$|E_1| < \left(\frac{\beta_1}{S}\right)^{1/2}, \quad \beta_1 = \frac{1}{\gamma M_1^2}\{1 + \gamma M_1^2 - M_1(\gamma+1)^{1/2} [2 + M_1^2(\gamma-1)(1-2\alpha)]^{1/2}\} \qquad (2.4)$$

the brances of curve L_2° intersect the u-axis with a vertical slope at points $u_{1,2}^\circ = (\gamma+1)^{-1}\{\gamma\Pi \pm [\gamma^2\Pi^2 - 2(\gamma^2-1)\Sigma]^{1/2}\}$, and one of the branches lies entirely in the subsonic and the other in the supersonic region.

When

$$|E_1| = \beta_1^{1/2} S^{-1/2} \qquad (2.5)$$

the two branches intersect the u-axis at one and the same point A which is the vertex of parabola L_5°

If, however,

$$|E_1| > (\beta_1 / S)^{1/2} \tag{2.6}$$

then one of the branches of curve L_2° lies entirely in the region $u > 0$, $E > 0$, and the second in the region $u > 0$, $E < 0$. In this case both branches have extrema at points

$$u_m = \left[\frac{2(\gamma-1)\Sigma}{\gamma+1}\right]^{1/2}, \quad E_m' = \pm\left\{\frac{1}{\gamma S}[2(\gamma^2-1)\Sigma]^{1/2} - \gamma\Pi\right\}^{1/2}$$

whch lie on parabola L_5°. The points of curve L_2° that lie to the right and to the left of parabola L_5° correspond, respectively, to supersonic and subsonic flows. If $M_1 > 1$, the inequalities $u_m < 1$, $|E_m| < |E_1|$ are satisfied.

Let us construct the curve $L_3^\circ(u, E, \varepsilon, \alpha)$. Since $\varepsilon \ll 1$, hence for $E \ll \varepsilon^{-1}$ the branches of curve L_3° must lie in small neighborhoods of lines L_1° and L_2°. Depending on the values of parameters M_1, γ, S and E_1 the relative position of curves L_1°, L_2° and L_4° can be different.

Several possible dispositions of these lines are shown in Figs. 1 and 4 – 7. For $J > 0$ line L_3° in the half-plane $u > 0$ consists of three branches. The intersection points of lines L_3° and L_4° are determined by a sixth order polynomial in E. Such points can exist in the half-plane $u > 0$ (Fig. 5), and there can be two (Figs. 1 and 6) or four (Fig. 4 and 7) of such points. These points, denoted by letters B, C, D and F in the Figures, are singular points of Eq. (2.3). A qualitative analysis of the integral curves of Eq. (2.3) shows that for $J > 0$ points B and D are focal points and points C and F are saddle points.

The points of intersection of lines L_1° and L_2° are determined by the cubic equation

$$Su(u^2 - E_1^2) - \frac{\gamma+1}{2\gamma}u^2 + \left(1 + \frac{1}{\gamma M_1^2}\right)u - \frac{1}{\gamma}\left[\frac{1}{M_1^2} + (\gamma-1)\left(\frac{1}{2} - \alpha\right)\right] = 0 \tag{2.7}$$

which was analyzed in [3] for $\alpha = 0$.

We introduce the notation

$$\beta_{2,3} = \frac{1}{(\gamma+1)M_1^2}\{1 + \gamma M_1^2 \pm [(M_1^2-1)^2 + 2\alpha(\gamma^2-1)M_1^4]^{1/2}\}$$

$$\beta_4 = \frac{\gamma M_1^2}{1+\gamma M_1^2}, \quad \beta_5 = \frac{\gamma M_1}{\{(\gamma+1)[2 + M_1^2(\gamma-1)(1-2\alpha)]\}^{1/2}}$$

$$\beta_6 = \frac{1}{4\beta_3^2\beta_4} - \left(\frac{1}{2\beta_3^4\beta_4^2} - \frac{1}{\beta_3^2}\right)^{1/2}, \quad \delta_1 = (S - \beta_4)/\beta_4 S^2$$

Note that for $\Sigma > 0$ and $\alpha > 0$ conditions $\beta_2 > 1$ and $\beta_3 > 0$ are satisfied. It can be shown that for

$$|E_1| < \beta_3 \tag{2.8}$$

and any S Eq. (2.7) has in region $u < \beta_2$ only one real root $u^{(1)}$ in the interval $|E_1| < u^{(1)} < \beta_3$, while for $\beta_3 < |E_1| < \beta_2$, depending on the value of S, it has in that region either one $u^{(1)}$ or three real roots $u^{(1)} < u^{(2)} \leqslant u^{(3)}$ in the interval $(\beta_3, |E_1|)$. Below we assume that the root $u^{(1)}$ lies in the subsonic region. If $|E_1| > \beta_2$, Eq. (2.7) has in region $u < \beta_2$ either no roots or has two real roots in the interval $\beta_3 < u^{(1)} \leqslant u^{(2)} < \beta_2$ (Fig. 7).

The branches of curve L_3° always pass through the small neighborhood of the intersection points of lines L_1° and L_2°.

If the inequalities

$$S > \beta_4, \quad |E_1| < \delta_1^{1/2} \tag{2.9}$$

are satisfied, lines L_1° and L_4° do not intersect, while for

$$S > \beta_4, \quad |E_1| > \delta_1^{1/2} \tag{2.10}$$

the straight line L_1° intersects parabola L_4° at two points. Since for $E_1^2 - \delta_1 \gg \varepsilon^2$ points C and F of the intersection of lines L_3° and L_4° lie in the neighborhood of these points, inequalities (2.10) may be taken as the approximate condition of the existence of singular points C and F of Eq.(2.3).

Let us, first, consider the case $J > 0$.

1°. Let the flow parameters be such that conditions (2.6) and (2.9) are simultaneously satisfied. It can be shown that for $S > \beta_5$ the inequalities $\delta_1 S > \beta_1$ and $S > \beta_4$ are simultaneously satisfied so that there exists the variation range of $|E_1|$ defined by inequalities (2.6) and (2.9).

The relative disposition of lines L_1°, L_2°, L_3°, L_4° and L_5° and the qualitative pattern of integral curves determined by Eq.(2.3) belonging to this case are shown in Fig. 1. The selection of the direction of motion along the integral curves corresponding to the downstream motion is determined by the sixth of Eqs.(1.1), which implies that for $q^* > 0$ the electric field always increases along the flow. This direction is shown in the diagrams by arrows. It is obvious that in the small neighborhood of branches of line L_3 where $L_3 \sim \varepsilon$, the tangent of the angle of inclination of integral curves is of the order of unity, while away from that line the slope of integral curves is small, since parameter $\varepsilon \ll 1$. At all points, except the singular ones, the slope of integral curves at intersection with line L_3° is vertical and at those with line L_4° it is horizontal. It should be noted, however, that at points of line L_4° Eq.(2.5) does not define the relation between the flow velocity and the electric field, since that equation was obtained for the condition that $\Pi + SE^2 - 2u \neq 0$.

When conditions (2.6) and (2.9) are satisfied, Eq.(2.3) has only two singular points in the half-plane $u > 0$, viz. B (a focal point) and C (a saddle point). Since these points lie in the ε-neighborhood of the intersection points of curves L_2° and L_4°, hence the coordinates of points B (u_b, E_b) and C (u_c, E_c) are determined to within smalls of order ε by the following formulas:

$$u_b = u_c = \left[\frac{2(\gamma-1)\Sigma}{3\gamma-1}\right]^{1/2}, \quad E_{b,c} = \pm\left(\frac{2u_b}{S} - \Pi\right)^{1/2} \tag{2.11}$$

It can be shown that when conditions (2.6) are satisfied, the radicand in the second of formulas (2.11) is positive. We use the notation

$$M_* = (3\gamma-1)^{1/2}\{\gamma(\gamma+1) - 2\gamma[(\gamma-1)^2 - 2\alpha(3\gamma-1)(\gamma-1)]^{1/2}\}^{-1/2}$$

Analysis of the second of formulas (2.11) shows that for $M_1 > M_*$ the inequality $|E_1| > |E_{b,c}|$ is satisfied, and for $1 < M_1 < M_*$ the inequality $|E_1| < |E_{b,c}|$.

Let us consider the integral curve that passes through point $(1, E_1)$, which corresponds to the initial state at point ζ_1 of the physical space. For $\varepsilon \to 0$ $(\lambda \to 0)$ the inital point evidently tends to the intersection point of lines $u = 1$ and L_2° $(u, E, 0)$.

a) Let $M_1 < M_*$ and $E_1 > 0$. The integral curve I (Fig. 1) after passing through the initial point $(1, E_1)$, moves to the small neighborhood of the upper branch of line L_3°, then intersects it and proceeds with a small negative slope toward smaller values of u. Then it enters the small neighborhood of the same branch of line L_3° and continues along it. The considered integral line cannot intersect line L_3° with a vertical tangent when $0 < E < E_b$ and $u < u_m$, since it has in that region a negative slope as the upper branch of line L_3° has for $u < u_m$.

To analyze the modes of flow in the physical plane, which correspond to various sections of the integral curve I we have to consider Eqs. (2.2), (1.4) and (2.3).

We denote by $\Delta\zeta_u$ and $\Delta\zeta_E$ the characteristic distances along which by virtue of Eqs. (2.2) and (1.4) the velocity and electric field variations are of order unity. On sections of the considered integral curve that lay along line L_3° and in its small neighborhood we have $|L_3| \sim \varepsilon$, $L_1 \sim 1$, $|L_4| \sim 1$, and $\Delta\zeta_u \sim 1$ and $\Delta\zeta_E \sim 1$. In the physical plane these sections correspond to regions in which the flow parameters (velocity, temperature, electric field and the charge density) vary and satisfy to within smalls of order ε the integrals of an ideal flow at constant electric potential, because in this case $|L_2| \sim \varepsilon$. The right-hand side section lies in the supersonic and the left-hand side one in the subsonic region.

Along the section of the integral curve I, which lies away from line L_3°, the quantities $|L_3|$, $|L_4|$ and L_1 are of the order of unity, and Eqs. (2.2) and (1.4) imply that there the lengths $\Delta\zeta_u \sim \varepsilon \ll 1$ and $\Delta\zeta_E \sim 1$. In the physical plane this section corresponds to the narrow region of order ε in which the velocity variation is considerable, while the electric field is nearly constant. This section of the integral curve defines the shock wave structure in a constant electric field. The part of the integral curve that is close to the point $(1, E_1)$ and runs along line L_3° defines the supersonic flow upstream of the wave front, and the part of that curve which runs along L_3° defines for $u < u_m$ the subsonic flow downstream of the wave. Neglecting the small variations of the electric field in the shock wave structure, from (1.2) and (1.5) we obtain to within terms of order ε the system of equations

$$\rho u = 1, \quad T = 1 + (1 - u)(\gamma M_1^2 u - 1) \tag{2.12}$$

$$\frac{dT}{d\zeta_*} = -\frac{(\gamma+1) M_1^2}{2}(u - 1)(u - \beta),$$

$$\beta = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M_1^2}, \quad \zeta_* = \frac{\zeta}{\varepsilon}$$

(β denotes the velocity downstream of the shock wave in an ideal flow) which defines the variation of parameters in the region of structure Γ.

The third of Eqs. (2.12) shows that the velocity variation in the wave structure from $u = 1$ to $u = \beta$ is accompanied by a monotonic increase of temperature from $T = 1$ to $T = T' \equiv 1 + 2(\gamma - 1)(M_1^2 - 1)(\gamma + M_1^{-2}) / (\gamma + 1)^2$. The curve of function $T = T(u)$ is shown in Fig. 2 (the section of parabola ac shown by the heavy line).

Thus in the considered case there exists for $1 < M_1 < M_*$ and $E_1 > 0$ a shock wave structure which, owing to the mechanism of thermal conductivity, is continuous in a constant electric field.

The interpretation of the integral curve II (Fig. 1) which for $M_1 < M_*$ and $E_1 < 0$ passes through point $(1, E_1)$ is similar.

For the sake of reducing the number of figures we represent below the integral curves which pass through the initial point $(1, E_1)$ for various values of parameters E_1, M_1, S and others in one and the same figure, assuming for convenience that to each case corresponds its own initial point.

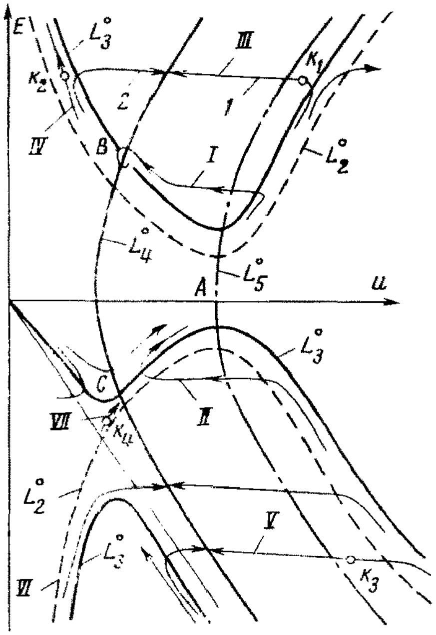

Fig. 1

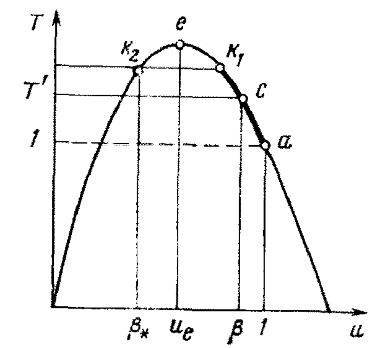

Fig. 2

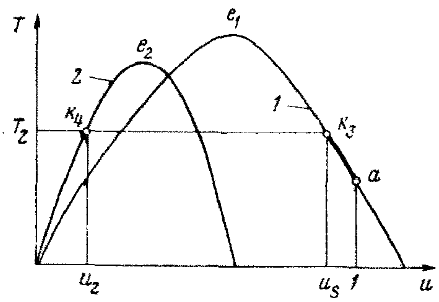

Fig. 3

b) Let us assume that $M > M_*$ and $E_1 > 0$, and consider the integral curve III (Fig. 1) which passes through the initial point $(1, E_1)$. Figure 1 shows that that integral curve has sections which run along line L_3° in the small neighborhood of the latter in both the super- and subsonic regions. At these sections the quantities $\Delta\zeta_u$ and $\Delta\zeta_E$ are of the order of unity, and the variation of parameters is close to that in an ideal (non-heat-conducting) flow. Sects. *1* and *2* of the integral curve III which lie away from line L_3° have, respectively, small negative and positive slopes. Along these sections $\Delta\zeta_u \sim \varepsilon$ and $\Delta\zeta_E \sim 1$. A narrow region Γ of a length of the order ε with considerable velocity variation and a constant electric field correspond to these sections in the physical field.

Let us assume that a continuous change of velocity from the initial $u = 1$ to the subsonic $u = \beta_*$ ($\beta_* = \beta$ for $M_1 > M_*$ in the last but one of formulas (2. 12)), which corresponds to the velocity downstream of the shock wave in an ideal flow, takes place in region Γ. In that region the variation of gasdynamic flow parameters in a constant electric field is defined by Eqs. (2. 12) and, contrary to the previous case, $\beta_* < u_e$, where $u_e = (2\gamma M_1)^{-2}(1 + \gamma M_1^2)$. The quantity u_e is equal to the velocity at the vertex of parabola $T = T(u)$ which links the velocity and temperature of flow in a constant electric field. Although Fig. 2 shows a drop of temperature with decreasing velocity after point e, the third of Eqs. (2. 11) implies that in region Γ, where $\beta_* < u < 1$, the temperature monotonically increases. This contradiction shows that under the considered conditions with $M_1 > M_*$ and $E_1 > 0$ and inequalities (2.6) and (2.9) satisfied, the thermal conductivity mechanism does not induce a continuous structure of the shock wave. This is also shown by the opposite direction of arrows in sections *1* and *2* of the integral curve III that lie to the right and left of line L_4° (Fig. 1).

The considered model thus shows that the gas can be changed over from the state defined by the initial point ζ_1 of the physical space to that at the final point ζ_4 only through a jump of gasdynamic parameters at some point inside region Γ. The continuous variation of parameters along the parabola between points a and k_1, where behind the shock wave front the gas temperature reaches its maximum equal T_2, corresponds in Fig. 2 to such transition. After that the velocity jump (and, consequently, of gas density and volume charge) moves at constant temperature from point k_1 to point k_2 where the velocity is close to the velocity downstream of the shock wave. In Fig. 1 this is shown by the continuous variation of parameters along the integral curve III up to point k_1, followed by a jump to point k_2 on some integral curve IV which runs along line L_3° (and along line L_2°) in its small neighborhood. At point k_1 the velocity is

$$u_s = \frac{2}{\gamma+1} - \frac{\gamma-1}{\gamma(\gamma+1)M_1^2}$$

Thus the discontinuous solution of system (1.2), (1.4) and (1.5) obtaining in this case is similar to that of the isothermic jump in conventional gasdynamics [4, 5]. Electrogasdynamic jumps at constant gas temperature were investigated in [3], where it is shown that when the charged particles move in the direction of the electric field, only jumps with continuous electric field exist.

The case of $E_1 < 0$, $M_1 > M_*$ and $|E_1| < u^{(1)}$, where $u^{(1)}$ is the root of Eq. (2.7) which determines the coordinates $(u^{(1)}, -u^{(1)})$ of the left-hand side intersection point of lines L_1° and L_2° is similar to the case considered in subsection (b).

c) Let $|E_1| > u^{(1)}$ and $E_1 < 0$. Let us consider the integral curve V (Fig. 1) which passes through the initial point $(1, E_1)$. As in the previous case that curve intersects line L_4° hence a continuous wave structure with constant electric field does not exist. It is evidently not possible to obtain in this case a discontinuous solution with constant field, since the integral curves that run along line L_2° in the subsonic region (of the kind VI, Fig. 1) lie in the region of negative electric current $J < 0$. It is shown in [3] that for $|E_1| > u^{(1)}$ when charged particles move against the direction of the electric field $E_1 < 0$ and $J > 0$, electrohydrodynamic jumps with a discontinuous field and constant gas temperature exist. The gas velocity and the electric field downstream are linked by the relationship

$$u_2 + E_2 = 0 \tag{2.13}$$

Let us investigate the feasibility of constructing discontinuous solutions of that kind using the considered model. Let the velocity and the field downstream of the wave satisfy formula (2.13). Velocity u_2 is determined by the cubic equation (2.7) and the temperature T_2 by the obtained values of velocity and field from the second of Eqs. (1.2). The velocity u_s at point k_3 of integral curve V, at which ends the continuous solution and an isothermic jump is present, is the root of the quadratic equation

$$u_s^2 - \left(1 + \frac{1}{\gamma M_1^2}\right) u_s + \frac{1}{\gamma M_1^2} T_2 = 0 \tag{2.14}$$

It is not difficult to establish the conditions under which (2.14) has a real root in the supersonic region

$$\left(1+\frac{1}{\gamma M_1^2}\right)^2 > \frac{4T_2}{\gamma M_1^2}, \quad \frac{\gamma M_1^2+1}{(\gamma+1)M_1^2} \leqslant u_s \leqslant 1 \qquad (2.15)$$

When conditions (2.15) are satisfied, it is possible to construct in this case a discontinuous solution with an isothermic jump for the input system of equations. Such solution is shown in Fig. 3. Between the initial point a and point k_3 of maximum temperature T_2, which corresponds to the temperature downstream of the shock, the velocity and the temperature constantly change along parabola 1 (the second of formulas (2.11)) with a constant field, and then the velocity (and also the electric field, the gas density, and the volume charge) changes abruptly at constant temperature to its value at point k_4 of parabola 2 (the second of formulas (1.2)) for $E = E_2 = -u_2$. This solution is represented in Fig. 1 by the segment of integral curve V between points $(1, E_1)$ to point k_3, then by the jump to point k_4 which lies near the subsonic point $(u^{(1)}, -u^{(1)})$ of the intersection of lines L_1° and L_2° on the integral curve VII. This is followed by the continuous variation of parameters that corresponds to an ideal flow along the integral curve VII which runs along L_3° in the ε-neighborhood of lines L_3° and L_2°.

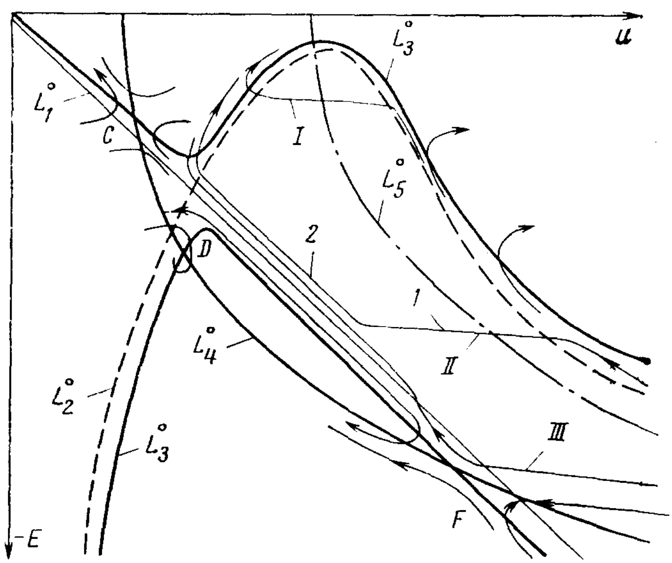

Fig. 4

Conditions (2.6), (2.9) and (2.15) together with inequalities $|E_1| > u^{(1)}$ and $E_1 < 0$, thus provide the necessary conditions of the existence of isothermic jumps with a varying electric field whose structure induced by the viscosity mechanism was investigated in [3].

2°. Let us investigate the case, when the flow parameters are such that the inequalities (2.6) and (2.10) are simultaneously satisfied, i.e.

$$\max\left\{\left(\frac{\beta_1}{S}\right)^{1/2}, \delta_1^{1/2}\right\} < |E_1|, \quad S > \beta_4$$

The disposition of lines L_3° and L_4° and the qualitative pattern of integral curve behavior in region $E > 0$ are similar to those considered in (a) and (b). In region $E < 0$ Eq. (2.3) has three singular points. The most interesting case of disposition of lines L_1°, L_2° and L_4° when the left-hand intersection point of lines L_1° and L_4° lies to the left of point $(u^{(1)}, -u^{(1)})$ of intersection of lines L_1° and L_2°, is shown in Fig. 4. The related criterion is of the form

$$u^{(1)} > S^{-1} - (E_1^2 - \delta_1)^{1/2}$$

If $|E_1| < \beta_3$ then, as noted above, $|E_1| < u^{(1)}$. The integral curve I which passes through the initial point $(1, E_1)$ defines a continuous shock wave structure with a constant electric field.

d) Let us consider in greater detail the integral curve II which for $\beta_3 < |E_1| < |E_F|$ passes through the initial point $(1, E_1)$. Note that E_F depends on E_1 and that for $E_1^2 - \delta_1 \gg \varepsilon^2$ point F lies in the ε-neighborhood of the right-hand point of intersection of parabola L_4° with the straight line L_1°. Allowing for this, we can represent the inequality $|E_1| < |E_F|$ to within quantities of order ε in the form $|E_1| > (2\beta_4)^{-1}$. The initial electric field is bounded by the condition $|E_1| < 1$, since we consider the case of $J > 0$. For $\gamma > 1$ and $M_1 > 1$ the quantity $\beta_4 > 0.5$. It can be shown that for $S > \beta_5$ conditions $\delta_1 S > \beta_1$ and $S > \beta_4$, and for $S > \beta_6$ the condition $\beta_3 > \delta_1^{1/2}$ are satisfied. Since $\beta_3 < 1$, there exists a range of variation of parameters S and M_1 for which the inequalities bounding from below the quantity $|E_1|$ do not violate the condition $J > 0$.

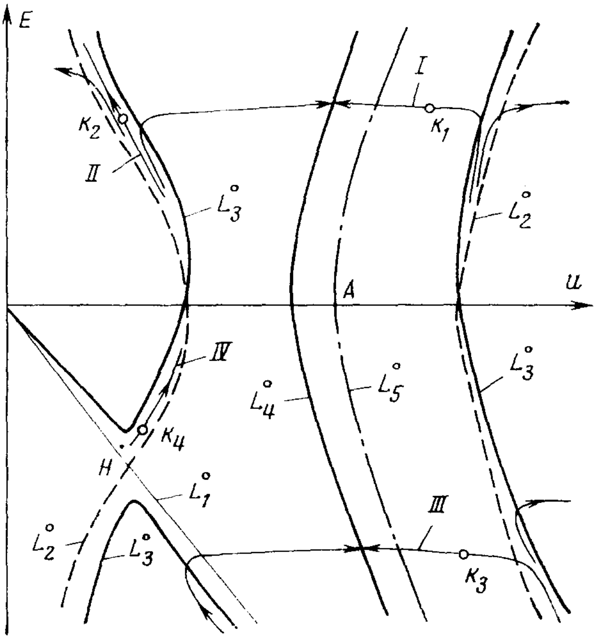

Fig. 5

If the root $u^{(1)}$ of Eq. (2.7) is in the subsonic region, the integral curve II has sections running near the line L_2° in the super- and subsonic regions. These sections correspond to the state of gas up- and downstream of the wave front. In the subsonic region the integral curve intersects line L_3° in the ε-neighborhood of line L_1°, where $u = -E$ and then runs along line L_3°. In the physical plane these sections correspond to regions of ideal (non-heat-conducting) flow, since here $|L_2| \sim \varepsilon$. Along section *1* of the considered integral curve we have $\Delta\zeta_u \sim \varepsilon$ and $\Delta\zeta_E \sim 1$. That section corresponds to the narrow flow region with considerable variation of velocity at constant electric field. Along section *2* the quantities $\Delta\zeta_u \sim \varepsilon$, $\Delta\zeta_E \sim \varepsilon$. This section describes a narrow flow region with a sharp variation of velocity and electric field. For $\varepsilon \to 0$ the velocity u_2 on this integral curve downstream of the shock wave is related to field E_2 by the equality $u_2 + E_2 = 0$ and is determined by the cubic equation (2.7).

The integral curve II thus defines a continuous structure of the shock wave induced by the thermal conductivity mechanism with a varying electric field. The structure shows that downstream of the wave the velocity and the electric field are related by the equality $u_2 + E_2 = 0$.

Note that the region of initial values of the electric field E_1 may be defined more precisely by analyzing the angles of the integral curve slope at intersection with L_1° in order to eliminate integral curves of type III which penetrate into the region of negative currents $J < 0$ and have no physical meaning.

The case when the initial electric field $|E_1| > |E_F|$ is similar to that considered in subsection (c). The inequality $|E_1| > |E_F|$ can be represented in the form $S^{-1} < |E_1| < (2\beta_4)^{-1}$ accurate to within quantities of order ε.

Note that when the root $u^{(1)}$ of Eq. (2.7) is in the supersonic region and the charged particles move in the opposite direction to the electric field ($J > 0$, $E_1 < 0$), the considered model does not indicate the existence of a shock wave structure.

3°. Let conditions (2.4) and (2.9) be simultaneously satisfied, which means that the electric field at point ζ_1 satisfies the condition

$$|E_1| < \min\left\{\left(\frac{\beta_1}{S}\right)^{1/2}, \delta_1^{1/2}\right\}, \quad S > \beta_4 \tag{2.16}$$

In this case Eq. (2.3) does not have singular points. The qualitative pattern of integral curves behavior is shown in Fig. 5.

It is obvious that in this case there are no initial conditions under which a continuous shock wave structure would exist. For $E_1 > -u^{(1)}$ integral curves of type I yield a structure with an isothermic jump and a continuous electric field.

If $E_1 < -u^{(1)}$, the electric field in the isothermic jump is discontinuous. The necessary conditions for the existence of such jumps are defined by the inequalities (2.15), (2, 16) and $|E_1| > u^{(1)}$. The last inequality and conditions (2.16) are not contradictory, if the parameters S, M_1 and γ are such that $\beta_3 < \min\{\delta_1^{1/2}, \beta_1^{1/2} S^{-1/2}\}$. This case is similar to that considered in subsection (c).

4°. It can be shown that for $\beta_4 < S < \beta_5$ conditions (2.4) and (2.10) are simultaneously satisfied, i.e. the case when

$$\delta_1^{1/2} < |E_1| < \beta_1^{1/2} S^{-1/2}$$

is possible.

Equation (2.3) has then two singular points in region $u > 0$, $E < 0$. The pattern of integral curves related to this case is shown in Fig. 6, where D and F are, respectively, focal and saddle points. The type of parameter variation in the shock wave structure is similar to that considered in case 3°, if the intersection point H of lines L_1° and L_2° lies to the left of the left intersection point of lines L_1° and L_4°, as in Fig. 6. When H lies to the right of point D, the qualitative pattern of behavior of integral curves is similar to the cases considered in case 2°.

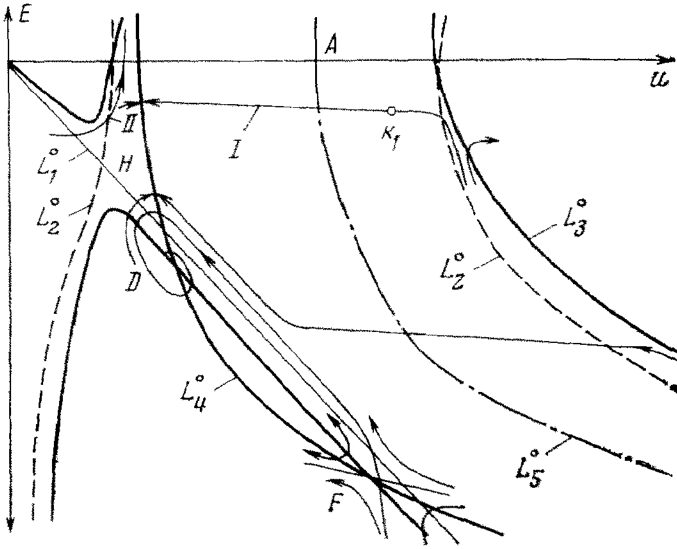

Fig. 6

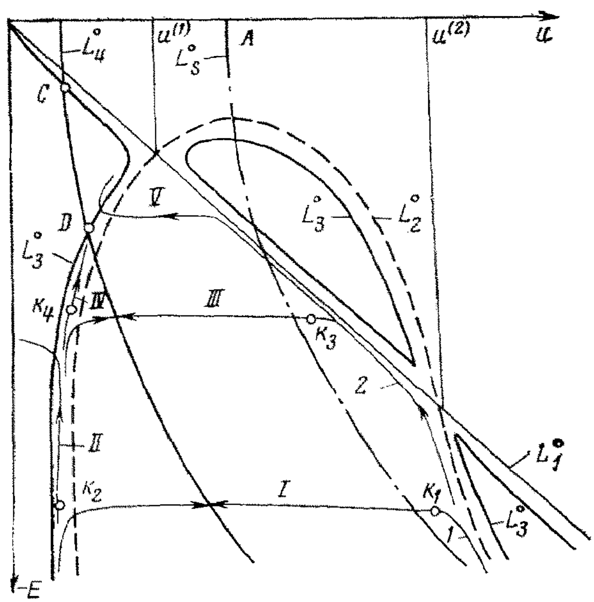

Fig. 7

3. Shock wave structure where the ions move in a direction opposite to that of gas. Let parameter $J < 0$ when $\Pi > 0$ and $\Sigma > 0$, and let Eq. (2.7) have three different roots $u^{(1)} < u^{(2)} < u^{(3)}$ to which correspond the values $E^{(1)} > E^{(2)} > E^{(3)}$ of the electric field (since the roots of (2.7) define one of the coordinates of intersection points of line L_2° with the straight line L_1°). The integral curves of Eq.(2.3) which have a physical meaning lie in the region $u > 0$, $E < 0$ below the straight line L_1° where $J < 0$. The qualitative pattern of behavior of integral curves is shown in Fig. 7 for the case in which condition (2.6) is satisfied, Eq.(2.3) has three singular points in region $u > 0$, $E < 0$, and the left-hand intersection point of lines L_1° and L_4° lies to the left of point $(u^{(1)}, E^{(1)})$ of intersection of lines L_1° and L_2°.

Let us consider several possible kinds of integral curves which yield a shock wave structure depending on the position of initial point $(1, E_1)$ which defines the state of gas upstream of the wave front. If $E^{(3)} < E_1 < E^{(2)}$, then the integral curve I defines the structure of a wave with an isothermic jump and a continuous electric field. The pattern is here similar to that considered in subsection (b). The wave structure consists of section *1* of the integral curve I and of the isothermic jump at constant field from point k_1 to point k_2 of the integral curve II. The velocity at point k_1 is determined from Fig. 2 for the obtained temperature T_2 downstream of the shock wave.

A distinctive feature of curves III and V is that the velocity and the electric field upstream of the wave front, whose structure is defined by any curve of this kind, are linked by the relationship $u^{*(2)} + bE^{*(2)} = 0$, where $u^{*(2)}$ and $E^{*(2)}$ are the dimensional coordinates of the mean intersection point of lines L_1° and L_2° (in dimensionless form $u = 1$, and $E = -1$). The electric field E_2 downstream of such wave front must be specified and, if $E_D < E_2 < E^{(1)}$, the shock wave structure is continuous with a variable electric field, produced by the thermal conductivity mechanism (curve V). When $E^{(2)} < E_2 < E_L$ and the conditions

$$f > \frac{4T_2}{\gamma M_1^2}, \quad \frac{\gamma}{\gamma+1} f \leqslant u_s \leqslant 1, \quad f = 1 + \frac{1}{\gamma M_1^2} + S(E_2^2 - 1) \tag{3.1}$$

are satisfied, there exist such waves with variable electric field whose structure consists of section *2* of the integral curve III, along which the electric field, the velocity and other parameters change abruptly, and of the isothermic jump with a constant electric field equal to that from point k_3 to some point k_4 of the integral curve IV. The position of point k_3 where the jump is located can be determined by resolving the conditions at the discontinuity which for a specified field downstream of the field are now closed. Then, using the second of formulas (1.2), the velocity u_s is determined at point k_3. The necessary conditions for the existence of such jumps are of the form (3.1).

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